COMPARATIVE ANALYSIS OF THE LAPLACE TRANSFORM AND THE FOURIER TRANSFORM FOR SOLVING SECOND-ORDER NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

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ABSTRACT.

This paper conducts a comparative analysis of two principal integral transform methods, the Laplace transform and the Fourier transform, for deriving analytical solutions to second-order linear non-homogeneous ordinary differential equations (ODEs). The study systematically examines their theoretical underpinnings, procedural methodologies, respective domains of applicability, and computational efficacy. Although both techniques reduce differential equations to more tractable algebraic forms, they exhibit fundamental distinctions in their operational domains, treatment of initial conditions, and suitability for various forcing functions.

The Laplace transform proves to be the more versatile and powerful tool for solving initial value problems (IVPs), which are prevalent in dynamical systems analysis. Its strength lies in its inherent capacity to incorporate initial conditions at $t = 0$ directly into the transformed equation and its robustness when dealing with discontinuous or exponential-type forcing functions.

Conversely, the Fourier transform is predominantly suited for problems defined on the entire real line$(-\infty, \infty)$. It emerges as the preferred method for steady-state analysis and for handling problems involving periodic or non-decaying signals, particularly those situated on infinite domains.

KEYWORDS: Laplace Transform, Fourier Transform, Differential Equations, Initial Value Problems, Transfer Function, Frequency Response, Integral Transforms.

1. INTRODUCTION.

Second-order ordinary differential equations (ODEs) are fundamental in modelling diverse physical systems, including mechanical vibrations, electrical circuits, and wave propagation. Although the Laplace and Fourier transforms both convert such differential equations into algebraic equations in a transformed domain, their respective domains of operation and scopes of applicability exhibit profound differences.

Laplace Transform (LT) [1, 2]**.**

-1- Solves initial value problems with transient responses.

- It converts differential equations into simpler algebraic equations in the s-domain, making them easier to solve.

- The solution inherently includes both the steady state and the complete transient response of the system.

-2- Handles discontinuous inputs (step functions, impulses).

- It provides a unified, algebraic framework for modeling inputs that are difficult to represent with standard calculus.
- This allows for the analysis of systems subjected to sudden shocks or changes, which are common in engineering.

-3- Used in control theory and circuit analysis.

- It is fundamental for deriving transfer functions, which model system dynamics and determine stability.
- Enables the analysis of complex circuit behavior (like RLC circuits) in the s-domain before converting back to time-domain results.

Fourier Transform (FT) [3, 4].

-1- Analyzes steady-state periodic systems

- It decomposes complex periodic signals into a sum of simple, infinite-duration sine and cosine waves.
- This is ideal for studying the long-term, stable behavior of systems driven by continuous oscillatory inputs.

-2- Provides direct frequency spectrum interpretation

- The transform's output directly shows the amplitude and phase of every frequency component present in a signal.
- This provides an intuitive and powerful visual representation of a signal's frequency content.

-3- Applied in signal processing and PDEs

- It is the cornerstone for operations like filtering, noise reduction, and audio/image compression.
- It is used to solve partial differential equations, such as the heat equation, by transforming spatial derivatives into algebraic terms.

2. THE ORETICAL FOUNDATIONS.

2.1 Laplace Transform.

Definition [1, 5].

Given a function $f(t)$ defined for $t \geq 0$, its Laplace transform is defined as.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Where: $s = \sigma + i\omega$ is a complex frequency parameter (with $\sigma, \omega \epsilon \mathbb{R}$), $f(t)$ is a piecewise continuous function of exponential order (i.e., $|f(t)| \leq Me^{at}$ for some $M, a > 0$).

Core characteristics.

1- Linearity. $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$.
2- Differentiation. $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.
3- Integration. $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$.
4- Time shifting. $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ (Where $u(t)$ is the Heaviside step function).
5- Frequency shifting. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

Inverse Laplace Transform.

The inverse operation, denoted $\mathcal{L}^{-1}$, recovers $f(t)$ from $F(s)$.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)\, e^{st} ds$$

Where γ is chosen such that all singularities of $F(s)$ lie to the left of the contour.

Applications.

- Solving ordinary and partial differential equations (ODEs/PDEs).
- Analysing control systems and electrical circuits.
- Studying signal processing and system stability.

The Laplace transform is particularly useful because it converts differential equations into algebraic equations, which are easier to manipulate and solve.

2.2 Fourier Transform.

Definition [3, 6].

Given a function $f(t)$ (where t is typically time), its Fourier transform $F(\omega)$ is defined as.

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

Where: ω is the angular frequency (in radians per second), i is the imaginary unit ($i^2 = -1$), $f(t)$ is an absolutely integral function (i.e., $\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$).

Inverse Fourier Transform.

The original function $f(t)$ can be recovered using the inverse Fourier transform.

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}d\omega$$

Core characteristics.

1. Linearity. $\mathcal{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$.
2. Differentiation. $\mathcal{F}\{f'(t)\} = i\omega F(\omega),\ \mathcal{F}\{f''(t)\} = -\omega^2 F(\omega)$.
3. Convolution Theorem.
   $\mathcal{F}\{(f * g)(t)\} = F(\omega).G(\omega), where\ (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$.
4. Time shifting. $\mathcal{F}\{f(t - t_0)\} = e^{-i\omega t_0}F(\omega)$.
5. Frequency shifting (Modulation). $\mathcal{F}\{e^{i\omega_0 t}f(t)\} = F(\omega - \omega_0)$.

Relation to Laplace Transform.

- The Fourier transform is a special case of the bilateral Laplace transform when $s = i\omega$.
- Unlike the Laplace transform, the Fourier transform is primarily used for analysing periodic and non-causal signals over the entire real line.

Applications.

- Signal processing (filtering, compression, noise removal).
- Image processing (JPEG compression, edge detection).
- Quantum mechanics (wave functions in momentum space).
- Solving PDEs (heat equation, wave equation).

The Fourier transform is fundamental in converting time-domain problems into frequency-domain representations, making it easier to analyse and manipulate signals and systems.

3. COMPARATIVE SOLUTIONS TO SECOND-ORDER ODES.

3.1 Solving a non-homogeneous differential equation with constant coefficients.

Example (1).

Solve the following differential equation $y''(t) + 3y'(t) + 2y(t) = f(t)$ With initial conditions $y(0) = y_0$, $y'(0) = v_0$ using the Laplace transform and the Fourier transform.

We will solve this using both the Laplace transform and the Fourier transform for concreteness.

Let's assume $f(t) = e^{-t}.u(t)$ $(where\ u(t) is\ the\ Heaviside\ step\ function)$, and the initial conditions are $y(0) = 0$, $y'(0) = 0$.

1. Solving Using the Laplace Transform Method.

The Laplace transform is well suited for initial value problems with causal forcing functions$(like\ f(t) = e^{-t}u(t))$.

Using the Laplace transform, we find that.

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s), \quad \mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y(t)\} = Y(s), \quad \mathcal{L}\{e^{-t}u(t)\} = \frac{1}{s+1}$$

Applying the Laplace transform to the differential equation.

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solve for Y(s). $\Longrightarrow$ $Y(s)(s^2 + 3s + 2) = \frac{1}{s+1}$

$$Y(s) = \frac{1}{(s+1)(s^2+3s+2)} = \frac{1}{(s+1)(s+1)(s+2)} = \frac{1}{(s+1)^2(s+2)}$$

Partial fraction decomposition, we express Y(s) as.

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Solving for A, B, C. $1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2$

$$let\ s = -1 \implies B = 1, \quad let\ s = -2 \implies C = 1$$

Now, expand and compare coefficients.

$$1 = A(s^2 + 3s + 2) + B(s+2) + C(s^2 + 2s + 1)$$
$$1 = (A + C)s^2 + (3A + B + 2C)s + (2A + 2B + C)$$

This gives. $A + C = 0 \implies A = -1$

$$3A + B + 2C = 0 \implies -3 + 1 + 2 = 0\ (consistent)$$
$$2A + 2B + C = 1 \implies -2 + 2 + 1 = 1\ (consistent)$$

Thus. $Y(s) = \frac{-1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2}$

Using known inverse transforms.

$$y(t) = -e^{-t} + te^{-t} + e^{-2t} = e^{-2t} + (t-1)e^{-t}, \qquad t \geq 0$$

2. Solving Using the Fourier Transform Method.

The Fourier transform is better suited for problems defined on $(-\infty, \infty)$ without initial conditions. However, we can adapt it for causal systems by assuming $y(t) = 0\ for\ t < 0$.

Using the Fourier transform, we find that.

$$\mathcal{F}\{y''(t)\} = (i\omega)^2Y(\omega) = -\omega^2Y(\omega), \quad \mathcal{F}\{y'(t)\} = i\omega Y(\omega)$$

$$\mathcal{F}\{y(t)\} = Y(\omega), \quad \mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+i\omega}$$

Applying the Fourier transform to the differential equation.

$$-\omega^2 Y(\omega) + 3i\omega Y(\omega) + 2Y(\omega) = \frac{1}{1+i\omega}$$

Solve for $Y(\omega)$.

$$Y(\omega)(-\omega^2 + 3i\omega + 2) = \frac{1}{1+i\omega}, \quad \Rightarrow Y(\omega) = \frac{1}{(1+i\omega)(-\omega^2+3i\omega+2)}$$

Factor the denominator. $-\omega^2 + 3i\omega + 2 = -(\omega^2 - 3i\omega - 2)$

The roots of $\omega^2 - 3i\omega - 2 = 0$ are.

$$\omega = \frac{3i \pm \sqrt{-9+8}}{2} = \frac{3i \pm i}{2} \quad \Rightarrow \quad \omega = 2i, \qquad \omega = i$$

Thus. $-\omega^2 + 3i\omega + 2 = -(\omega - 2i)(\omega - i) = (i\omega + 2)(i\omega + 1)$

So $Y(\omega) = \frac{1}{(1+i\omega)(1+i\omega)(2+i\omega)} = \frac{1}{(1+i\omega)^2(2+i\omega)}$

Partial fraction decomposition, express $Y(\omega)$ as: $Y(\omega) = \frac{A}{1+i\omega} + \frac{B}{(1+i\omega)^2} + \frac{C}{2+i\omega}$

Solving for A, B, C.

$$let\ i\omega = -1 \quad \Rightarrow B = 1, \ let\ i\omega = -2 \quad \Rightarrow C = 1$$

Now, expand and compare coefficients.

$$1 = A(2 + 3i\omega + (i\omega)^2) + B(2 + i\omega) + C(1 + 2i\omega + (i\omega)^2)$$
$$1 = (A + C)(i\omega)^2 + (3A + B + 2C)(i\omega) + (2A + 2B + C)$$

This gives. $A + C = 0 \ \Rightarrow A = -1$

$$3A + B + 2C = 0 \ \Rightarrow \ -3 + 1 + 2 = 0\ (consistent)$$
$$2A + 2B + C = 1 \ \Rightarrow \ -2 + 2 + 1 = 1\ (consistent)$$

Thus. $Y(\omega) = \frac{-1}{1+i\omega} + \frac{1}{(1+i\omega)^2} + \frac{1}{2+i\omega}$

Using known inverse transforms.

$$\mathcal{F}^{-1}\left\{\frac{1}{(1+i\omega)^n}\right\} = \frac{t^{n-1}e^{-t}}{(n-1)!}u(t), \qquad \mathcal{F}^{-1}\left\{\frac{1}{1+i\omega}\right\} = e^{-t}u(t)$$
$$\mathcal{F}^{-1}\left\{\frac{1}{(1+i\omega)^2}\right\} = te^{-t}u(t), \quad \mathcal{F}^{-1}\left\{\frac{1}{2+i\omega}\right\} = e^{-2t}u(t)$$

Thus.

$$y(t) = -e^{-t}u(t) + te^{-t}u(t) + e^{-2t}u(t) = [e^{-2t} + (t-1)e^{-t}]u(t)$$

This matches the solution obtained via the Laplace transform for $t \geq 0$.

3.2 Solve the non-homogeneous differential equation in the general form.

Example (2).

Compare the solution of the general form differential equation $ay'' + by' + cy = x(t)$ using the Laplace transform and the Fourier transform.

1. Solving Using the Laplace Transform Method [1, 7].

A). Take the Laplace transform of both sides of the equation:

$$\mathcal{L}\{ay''(t) + by'(t) + cy(t)\} = \mathcal{L}\{x(t)\}$$

Using the differentiation properties of the Laplace transform.

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = X(s)$$

$$where\ Y(s) = \mathcal{L}\{y(t)\} and\ X(t) = \mathcal{L}\{x(t)\}$$

B). Solve for Y(s). $Y(s) = \frac{X(s)+asy(0)+ay'(0)+by(0)}{as^2+bs+c}$

This gives the Laplace-domain solution.

C). Take the inverse Laplace transform to find $y(t)$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

main points.

* The Laplace transform incorporates initial conditions $y(0) and\ y'(0)$.

* It is well-suited for causal systems $(where\ t \geq 0)$.

* The method works for a broader class of functions, including those that are not absolutely integrable (e.g., exponential and polynomial growth).

2. Solving Using the Fourier Transform Method [3, 8].

The Fourier transform is typically used for problems defined over $t\epsilon(-\infty, \infty)$, especially for steady-state or frequency-domain analysis.

A). Take the Fourier transform of both sides.

$$\mathcal{F}\{ay''(t) + by'(t) + cy(t)\} = \mathcal{F}\{x(t)\}$$

Using the differentiation properties of the Fourier transform.

$$a(i\omega)^2 Y(\omega) + b(i\omega)Y(\omega) + cY(\omega) = X(\omega)$$

$$where\ Y(\omega) = \mathcal{F}\{y(t)\} and\ X(\omega) = \mathcal{F}\{x(t)\}$$

B). Solve for $Y(\omega)$. $Y(\omega) = \frac{X(\omega)}{a(i\omega)^2+b(i\omega)+c} = \frac{X(\omega)}{-a\omega^2+ib\omega+c}$

This gives the frequency-domain solution.

$$y(t) = \mathcal{F}^{-1}\{Y(s)\}$$

main points.

- The Fourier transform does not incorporate initial conditions directly; it assumes the system is in steady state or defined over all time.
- It requires $x(t) and\ y(t)$ to be absolutely integrable (or square integrable), which restricts its use for certain functions (e.g., growing exponentials).
- It is particularly useful for analysing frequency response and signal processing applications.

4. CASE STUDIES.

4.1 Damped Harmonic Oscillator.

Example (3)

Solve the differential equation. $m\ddot{x} + c\dot{x} + kx = F_0 Cos(\omega_d t)$.

1. Laplace Transform mothed [1, 9].

- Take the Laplace transform of both sides, assuming zero initial conditions $(x(0) = 0, \dot{x}(0) = 0)$.

$$(ms^2 + cs + k)X(s) = F_0 \frac{s}{s^2 + \omega_d^2}$$

- Solve for $X(s)$. $X(s) = \frac{F_0 s}{(ms^2+cs+k)(s^2+\omega_d^2)}$

- Perform partial fraction decomposition to separate the transient and steady-state components.

- The steady-state solution (arising from the poles at $s = \pm i\omega_d$) is:

$$x_{ss}(t) = Re\left[\frac{F_0 e^{i\omega_d t}}{k - m\omega_d^2 + ic\omega_d}\right]$$

This simplifies to.

$$x_{ss}(t) = \frac{F_0}{\sqrt{(k-m\omega_d^2)^2+(c\omega_d)^2}}\cos(\omega_d t - \emptyset), \quad \text{where} \quad \emptyset = tan^{-1}\left(\frac{c\omega_d}{k-m\omega_d^2}\right).$$

2. Fourier Transform mothed [3, 10].

- Assume a steady-state solution of the form $x(t) = Xe^{i\omega_d t}$.

- Substitute into the differential equation to solve for the complex amplitude X.

$$X = \frac{F_0}{k - m\omega_d^2 + ic\omega_d}$$

- The magnitude and phase give the same steady-state solution as the Laplace method.

3. Transient Solution (Homogeneous Equation).

- The transient response (from $m\ddot{x} + c\dot{x} + kx = 0$) is.

$$x_h(t) = e^{-\zeta\omega_n t}(Acos(\omega_d' t) + Bsin(\omega_d' t))$$

Where. $\omega_n = \sqrt{\frac{k}{m}}$ (Natural frequency), $\zeta = \frac{c}{2\sqrt{mk}}$ (damping ratio), $\omega_d' = \omega_n\sqrt{1-\zeta^2}$ (damped natural frequency).

Final Solution.

- The total solution is the sum of transient and steady-state responses. $x(t) = x_h(t) + x_{ss}(t)$

- Over time, the transient term decays $(x_h(t) \to 0)$, leaving only the steady-state oscillation.

Key Observations on the Result.

1. Steady-State Amplitude and Phase.

- The amplitude of the steady-state response depends on.

* The driving frequency ω_d relative to the natural frequency ω_n.

* The damping coefficient c.

- Resonance occurs when $\omega_d \approx \omega_n$ (if damping is small), leading to large oscillations.

- The phase lag $\emptyset$ indicates how much the response lags behind the driving force.

2. Effect of Damping.

- Higher damping (c) reduces the amplitude and smoothens the resonance peak.

- Critical damping $(\zeta = 1)$ eliminates oscillations in the transient response.

3. Transient vs. Steady State.

- The transient response depends on initial conditions but decays exponentially.

- The steady-state response persists indefinitely and dominates the long-term behaviour.

Final Answer (Steady-State Solution).

The steady-state displacement of the system is:

$$x_{ss}(t) = \frac{F_0}{\sqrt{(k - m\omega_d^2)^2 + (c\omega_d)^2}} \cos(\omega_d t - \emptyset), \quad \emptyset = tan^{-1}(\frac{c\omega_d}{k - m\omega_d^2})$$

This represents a forced oscillation at the driving frequency ω_d, with amplitude and phase determined by the system's parameters. The solution is consistent across both Laplace and Fourier transform methods, with the Laplace method additionally capturing transient effects.

4.2 RLC Circuit Analysis.

Example (4).

Solve the differential equation. $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = V(t)$.

When solving the differential equation using the Laplace transform and the Fourier transform, we obtain the following results.

* Laplace transform solution [2, 7].

$$I(t) = \mathcal{L}^{-1}\left\{\frac{\tilde{V}(s) + LsI(0) + LI'(0) + RI(0)}{Ls^2 + Rs + \frac{1}{C}}\right\}$$

This includes initial condition and is suited for initial value problems.

* Fourier transform solution [4, 8].

$$I(t) = \mathcal{F}^{-1}\left\{\frac{\hat{V}(\omega}{\frac{1}{C} - L\omega^2 + iR\omega}\right\}$$

This is a convolution with the impulse response $G(t)$, and is suited for problems defined on $-\infty < t < \infty$.

Both methods yield equivalent results for the forced response, but Laplace is better for incorporating initial conditions.

5. DISCUSSION.

Criterion	Laplace Transform	Fourier Transform
Domain of Operation	Complex s-plane ($s = \sigma + i\omega$)	Pure imaginary frequency axis ($i\omega$)
Initial Conditions	Directly incorporated via derivative properties	Not inherently equipped for initial value problems
System Stability	Robustly handles unstable systems via ROC analysis	Requires absolute integrability for direct convergence
Transient Response	Explicitly resolves transient and steady-state components	Inherently captures global frequency content
Handling	Native support for singularities and step	Requires distribution theory for

Discontinuities	inputs	rigorous treatment
Ideal for Signal Type	Causal signals defined for $\boldsymbol{t \geq 0}$	Energy signals existing on $(-\infty, \infty)$
Mathematical Foundation	Defined via one-sided integration kernel e^{-st}	Defined via two-sided oscillation kernel $e^{-i\omega t}$
Primary Application Context	Initial Value Problems in dynamics and control	Boundary Value Problems and steady-state analysis

6. CONCLUSION.

- Laplace transform is natural and optimal for time-domain IVPs for initial value problems with $t \geq 0$.
- Fourier transform excels in frequency response analysis, and Fourier Transform can also be used, especially if we interpret the system as causal $(i.e., y(t) = 0 \; for \; t < 0)$.
- Both methods yield the same solution for t≥0, but the Fourier transform requires careful handling of the region $t < 0$, and Choice depends on system stability and solution objectives.

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